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A Stronger LP Bound for Formula Size Lower Bounds via Clique Constraints[☆]

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Abstract

We introduce a new technique proving formula size lower bounds based on the linear programming bound originally introduced by Karchmer, Kushilevitz and Nisan and the theory of stable set polytopes. We apply it to majority functions and prove their formula size lower bounds improved from the classical result of Khrapchenko. Moreover, we introduce a notion of unbalanced recursive ternary majority functions motivated by a decomposition theory of monotone self-dual functions and give matching upper and lower bounds of their formula size. We also show monotone formula size lower bounds of balanced recursive ternary majority functions improved from the quantum adversary bound of Laplante, Lee and Szegedy.

Keywords: Formula Size Lower Bound, Linear Programming, Monotone Self-Dual Boolean Function, Stable Set Polytope

1. Introduction

Proving formula size lower bounds is a fundamental problem in complexity theory and also an extremely tough problem to resolve. A super-polynomial lower bound of a function in \mathbf{NP} implies $\mathbf{NC}^1 \neq \mathbf{NP}$ [29]. There are a lot of techniques to prove formula size lower bounds, e.g. [13, 25, 14, 11, 18, 7, 15, 16, 8]. Laplante, Lee and Szegedy [15] introduced a technique based on the quantum adversary method [1] and gave a comparison with known techniques. In particular, they showed that their technique subsumes several known techniques such as Khrapchenko [13] and its extension [14]. The current best formula size lower bound is $n^{3-o(1)}$ by Håstad [7] and a key lemma used in the proof is also

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subsumed by the quantum adversary bound [15]. Karchmer, Kushilevitz and Nisan [11] introduced a technique proving formula size lower bounds called the LP (linear programming) bound and showed that it cannot prove a lower bound larger than $4n^2$ for non-monotone formula size in general. Lee [16] proved that the LP bound [11] subsumes the quantum adversary bound [15] and Høyer, Lee and Špalek [8] introduced a stronger version of the quantum adversary bound.

Motivated by the result of Lee [16], we devise a stronger version of the LP bound by using an idea from the theory of stable set polytopes, known as clique constraints [20]. Suggesting a stronger technique compared to the original LP bound [11] has possibilities to improve the best formula size lower bound because it subsumes many techniques including the key lemma of Håstad [7]. Moreover, our technique has various possibilities of extensions such as rank constraints [21] discussed in Section 6 and orthonormal constraints [6], each of which subsume clique constraints. Due to this extendability, it is difficult to show the limitation of our new technique.

To study the relative strength of our technique, we apply it to some families of Boolean functions. For each family, we have distinct motivation to investigate their formula size. Three kinds of Boolean functions treated in this paper are defined as follows. They are monotone functions invariant under negations of all the input variables and the output (i.e., self-dual).

Definition 1.1. A majority function $\text{MAJ}_{2l+1} : \{0, 1\}^{2l+1} \mapsto \{0, 1\}$ outputs 1 if the number of 1's in the input bits is greater than or equal to $l + 1$ and 0 otherwise. We define unbalanced recursive ternary majority functions $\text{URecMAJ}_3^h : \{0, 1\}^{2h+1} \mapsto \{0, 1\}$ as

$$\text{URecMAJ}_3^h(x_1, \dots, x_{2h+1}) = \text{MAJ}_3(\text{URecMAJ}_3^{h-1}(x_1, \dots, x_{2h-1}), x_{2h}, x_{2h+1})$$

with $\text{URecMAJ}_3^1 = \text{MAJ}_3$. We also define balanced recursive ternary majority functions $\text{BRecMAJ}_3^h : \{0, 1\}^{3^h} \mapsto \{0, 1\}$ as

$$\begin{aligned} \text{BRecMAJ}_3^h(x_1, \dots, x_{3^h}) &= \text{MAJ}_3(\text{BRecMAJ}_3^{h-1}(x_1, \dots, x_{3^{h-1}}), \\ &\quad \text{BRecMAJ}_3^{h-1}(x_{3^{h-1}+1}, \dots, x_{2 \cdot 3^{h-1}}), \\ &\quad \text{BRecMAJ}_3^{h-1}(x_{2 \cdot 3^{h-1}+1}, \dots, x_{3^h})) \end{aligned}$$

with $\text{BRecMAJ}_3^1 = \text{MAJ}_3$. Throughout the paper, n means the number of input bits. Formula size and monotone formula size of a Boolean function f are denoted by $L(f)$ and $L_m(f)$, respectively.

Although our improvements of lower bounds seem to be slight, it breaks a stiff barrier (known as the certificate complexity barrier [15]) of previously known proof techniques. The best monotone upper and lower bounds of majority functions are $O(n^{5.3})$ [30] and $\lfloor n/4 \rfloor n(1 + \log \frac{n}{n-2})$ [24], respectively. In the non-monotone case, the best formula size upper and lower bounds of majority functions are $O(n^{4.57})$ [22] and $\lceil n/2 \rceil^2 (= (l+1)^2 \text{ when } n = 2l+1)$, respectively, which can be proven by the classical result of Khrapchenko [13]. In this paper,

we slightly improve the non-monotone formula size lower bound while no previously known techniques has been able to improve it since 1971. In Section 4, we will prove

$$L(\mathbf{MAJ}_{2l+1}) \geq (l+1)^2 + 1.$$

It is known that the class of monotone self-dual Boolean functions is closed under compositions (equivalently, in Post's lattice [5, 23]). Any monotone self-dual Boolean function can be decomposed into compositions of 3-bit majority functions [9]. A key observation for our proofs is that a communication matrix (defined in the next section) of a monotone self-dual Boolean function contains those of the 3-bit majority function as its submatrices.

Ibaraki and Kameda [9] developed a decomposition theory of monotone self-dual Boolean functions in the context of mutual exclusion in distributed systems. The theory has been further investigated in [3, 4]. They showed that any monotone self-dual function shares the structure of $\mathbf{URecMAJ}_3^h$ in the following sense. Let f be a monotone self-dual Boolean function and g be the function $g(x_2, \dots, x_n) = f(0, x_2, \dots, x_n)$. We can decompose g as $g = f_1 \wedge f_2 \wedge \dots \wedge f_k$. Then, f can be written as

$$f = \mathbf{MAJ}_3(x_1, f_1, (\mathbf{MAJ}_3(x_1, f_2, \mathbf{MAJ}_3(\dots \mathbf{MAJ}_3(x_1, f_{k-1}, f_k))))).$$

It holds $\mathbf{URecMAJ}_3^h$ in its internal structure.

To determine its formula size is of particular interest because $\mathbf{URecMAJ}_3^h$ can be regarded as one of the most basic cases among all the monotone self-dual Boolean functions. Its analysis would be helpful for analysis for any monotone self-dual Boolean function in general as a first step. In Section 5, we will prove

$$L(\mathbf{URecMAJ}_3^h) = L_m(\mathbf{URecMAJ}_3^h) = 4h + 1.$$

Balanced recursive ternary majority functions have been studied in several contexts [10, 15, 17, 19, 27, 28], see [15] and [27] for details. Ambainis et al. [2] showed a quantum algorithm which evaluates a monotone formula (or called AND-OR formula) of size N in $N^{1/2+o(1)}$ query even if it is not balanced. This result implies $\mathbf{BRecMAJ}_3^h$ can be evaluated in $O(\sqrt{5}^h)$ query by the quantum algorithm because we have a formula size upper bound

$$L_m(\mathbf{BRecMAJ}_3^h) \leq 5^h$$

as noted in [15]. Improving this result, Reichardt and Spalek [27] gave a quantum algorithm which evaluates $\mathbf{BRecMAJ}_3^h$ in $O(2^h)$ query. Reichardt [26] has now shown that any formula of size N can be evaluated by a quantum algorithm in $O(\sqrt{N})$ query. From this context, seeking the true bound of the monotone formula size of $\mathbf{BRecMAJ}_3^h$ is a very interesting research question.

The quantum adversary bound [15] has a quite nice property written as

$$\mathbf{ADV}(f \cdot g) \geq \mathbf{ADV}(f) \cdot \mathbf{ADV}(g).$$

It directly implies a formula size lower bound

$$L(\mathbf{BRecMAJ}_3^h) \geq 4^h.$$

The generalized adversary bound of Høyer, Lee and Špalek [8] has the same property. It has been revealed that it exactly characterizes the quantum query complexity [26].

In Section 6, we will prove

$$L_m(\mathbf{BRecMAJ}_3^2) \geq 20$$

and

$$L_m(\mathbf{BRecMAJ}_3^h) \geq 4^h + \frac{13}{36} \cdot \left(\frac{8}{3}\right)^h.$$

This gives a slight improvement of the lower bound and means that the 4^h lower bound is at least not optimal in the monotone case.

2. Preliminaries

We define a total order $0 < 1$ between the two Boolean values. For Boolean vectors $\vec{x} = (x_1, \dots, x_n)$ and $\vec{y} = (y_1, \dots, y_n)$, we define $\vec{x} \leq \vec{y}$ if $x_i \leq y_i$ for all $i \in \{1, \dots, n\}$. A Boolean function f is called monotone if $\vec{x} \leq \vec{y}$ implies $f(\vec{x}) \leq f(\vec{y})$ for all $\vec{x}, \vec{y} \in \{0, 1\}^n$. For a monotone Boolean function f , a Boolean vector $\vec{x} \in \{0, 1\}^n$ is called minterm if $f(\vec{x}) = 1$ and $(\vec{y} \leq \vec{x}) \wedge (\vec{x} \neq \vec{y})$ implies $f(\vec{y}) = 0$ for any $\vec{y} \in \{0, 1\}^n$ and called maxterm if $f(\vec{x}) = 0$ and $(\vec{x} \leq \vec{y}) \wedge (\vec{x} \neq \vec{y})$ implies $f(\vec{y}) = 1$ for any $\vec{y} \in \{0, 1\}^n$. Sets of all minterms and maxterms of a monotone Boolean function f are denoted by $\min T(f)$ and $\max T(f)$, respectively. A Boolean function f is called self-dual if $f(x_1, \dots, x_n) = f(\bar{x}_1, \dots, \bar{x}_n)$ where \bar{x} is the negation of x . Remark that if a Boolean function f is self-dual, its communication matrix (see below) has some nice properties, e.g. $|X| = |Y|$.

A formula is a binary tree with leaves labeled by literals and internal nodes labeled by \wedge and \vee . A literal is either a variable or the negation of a variable. A formula is called monotone if it does not have negations. It is known that all (monotone) Boolean functions can be represented by a (monotone) formula. The size of a formula is its number of leaves. We define the (monotone) formula size of a Boolean function f as the size of the smallest (monotone) formula computing f .

Karchmer and Wigderson [12] characterize formula size of any Boolean function in terms of a communication game called the Karchmer-Wigderson game. In the game, given a Boolean function f , Alice gets an input \vec{x} such that $f(\vec{x}) = 1$ and Bob gets an input \vec{y} such that $f(\vec{y}) = 0$. The goal of the game is to find an index i such that $x_i \neq y_i$. They also characterize monotone formula size by a monotone version of the Karchmer-Wigderson game. In the monotone game, Alice gets a minterm \vec{x} and Bob gets a maxterm \vec{y} . The goal of the monotone game is to find an index i such that $x_i = 1$ and $y_i = 0$. The number of leaves in a best communication protocol for the (monotone) Karchmer-Wigderson game is equal to the (monotone) formula size of f . From these characterizations, we consider communication matrices derived from the games.

Definition 2.1 (Communication Matrix). *Given a Boolean function f , we define its communication matrix as a matrix whose rows and columns are indexed by $X = f^{-1}(1)$ and $Y = f^{-1}(0)$, respectively. Each cell of the matrix contains indices i such that $x_i \neq y_i$. In a monotone case, given a monotone Boolean function f , we define its monotone communication matrix as a matrix whose rows and columns are indexed by $X = \min T(f)$ and $Y = \max T(f)$, respectively. Each cell of the matrix contains indices i such that $x_i = 1$ and $y_i = 0$. A combinatorial rectangle is a direct product $X' \times Y'$ where $X' \subseteq X$ and $Y' \subseteq Y$. A combinatorial rectangle $X' \times Y'$ is called monochromatic with respect to f if every cell $(\vec{x}, \vec{y}) \in X' \times Y'$ contains the same index i . We call a cell a singleton if it contains just one index.*

The minimum number of disjoint monochromatic rectangles which exactly cover all cells in the (monotone) communication matrix gives a lower bound for the number of leaves of a best communication protocol for the (monotone) Karchmer-Wigderson game. Thus, we obtain the following bound.

Theorem 2.2 (Rectangle Bound [12]). *The minimum size of an exact cover by disjoint monochromatic rectangles for the communication matrix (or monotone communication matrix) associated with a Boolean function f gives a lower bound of $L(f)$ (or $L_m(f)$).*

3. A Stronger Linear Programming Bound via Clique Constraints

In this study, we devise a new technique proving formula size lower bounds based on the LP bound [11] with clique constraints. We assume that readers are familiar with the basics of linear and integer programming theory. Karchmer, Kushilevitz and Nisan [11] formulate the rectangle bound as an integer programming problem and give its LP relaxation.

Let R be the set of all monochromatic rectangles and x_r be a variable associated with a monochromatic rectangle $r \in R$. Given a (monotone) communication matrix, it can be written as

$$\begin{aligned} \min \quad & \sum_{r \in R} x_r \quad (\text{number of rectangles}) \\ \text{s.t.} \quad & \sum_{r \ni c} x_r = 1, \quad (\text{for each cell } c \text{ in the matrix}) \\ & x_r \geq 0. \quad (\text{for each monochromatic rectangle } r \in R) \end{aligned}$$

The dual problem can be written as

$$\begin{aligned} \max \quad & \sum_c w_c \quad (\text{sum of weights}) \\ \text{s.t.} \quad & \sum_{c \in r} w_c \leq 1. \quad (\text{for each monochromatic rectangle } r \in R) \end{aligned}$$

Here, each variable w_c is indexed by a cell c in the matrix. From the duality theorem, showing a feasible solution of the dual problem gives a formula size lower bound.

Now, we introduce our stronger LP bound using clique constraints from the theory of stable set polytopes. We assume that each monochromatic rectangle is a node of a graph. We connect two nodes by an edge if the two corresponding monochromatic rectangles intersect. If a set of monochromatic rectangles q compose a clique in the graph, we add a constraint

$$\sum_{r \in q} x_r \leq 1$$

to the primal problem of the LP relaxation. This constraint is valid for all integral solutions since we consider the disjoint cover problem. That is, we can assign the value 1 to at most 1 rectangle in a clique for all integral solutions under the condition of disjointness.

The dual problem can be written as

$$\begin{aligned} \max \quad & \sum_c w_c + \sum_q z_q \\ \text{s.t.} \quad & \sum_{c \in r} w_c + \sum_{q \ni r} z_q \leq 1, \quad (\text{for each monochromatic rectangle } r \in R) \\ & z_q \leq 0. \quad (\text{for each clique } q) \end{aligned}$$

Intuitively, this formulation can be interpreted as follows. Each cell c is assigned a weight w_c . The summation of weights over all cells in a monochromatic rectangle is limited to 1. This limit is decreased by 1 if it is contained in a clique. Thus, the limit of the total weight for a monochromatic rectangle contained in k distinct cliques is $k + 1$.

In our proofs, we utilize the following property of combinatorial rectangles which is trivial from the definition. If a rectangle contains two cells (α_1, β_1) and (α_2, β_2) , it also contains both (α_1, β_2) and (α_2, β_1) . A notion of singleton cells also occupies an important role for our proofs because there are no monochromatic rectangles which contain different kinds of singleton cells.

4. Formula Size of Majority Functions

In this section, we show a non-monotone formula size lower bound of the majority function improved from the classical result of Khrapchenko [13]. First, we look at the case for the 3-bit majority function. The original LP bound can show a lower bound of at most 4.5, as can be easily checked by giving an upper bound to the primal problem. As formula size must be an integer, this already implies a lower bound of 5 on the formula size of the 3-bit majority function. By using clique constraints, however, we can go beyond this limitation and directly show the exact lower bound of 5.

Proposition 4.1.

$$L(\text{MAJ}_3) = L_m(\text{MAJ}_3) = 5.$$

Proof. We have a monotone formula $(x_1 \wedge x_2) \vee ((x_1 \vee x_2) \wedge x_3)$ for \mathbf{MAJ}_3 . From the definition, $L(\mathbf{MAJ}_3) \leq L_m(\mathbf{MAJ}_3)$. To prove $L(\mathbf{MAJ}_3) \geq 5$, we consider a communication matrix of the 3-bit majority function whose rows and columns are restricted to minterms and maxterms, respectively. There are three 2×2 monochromatic rectangles for each three index and these form a clique as Figure 2.

	100	010	001
110	2	1	1,2,3
101	3	1,2,3	1
011	1,2,3	3	2

Figure 1: The Communication Matrix of \mathbf{MAJ}_3 . (Rows are 1-inputs and columns are 0-inputs. Cells indicate where the inputs differ.)

In the dual problem, we assign weights 1 for all singleton cells and 0 for other cells. There are 6 singleton cells and hence the total weight is 6. We take the clique q composed of the three 2×2 monochromatic rectangles containing two singleton cells. It is clear that every pair of monochromatic rectangles contained in q intersect at some cell. We assign $z_q = -1$. Then, the objective function of the dual problem becomes $5 = 6 - 1$.

	100	010	001
110		1	1,2,3
101		1,2,3	1
011			

	100	010	001
110	2		1,2,3
101			
011	1,2,3		2

	100	010	001
110			
101	3	1,2,3	
011	1,2,3	3	

Figure 2: Three Monochromatic Rectangles which Form a Clique q (K_3)

Now, we show that all constraints of the dual problem are satisfied. First, we consider a monochromatic rectangle which contains at most one singleton cell. In this case, the constraint is clearly satisfied because the summation of weights in the monochromatic rectangle is less than or equal to 1. Then, we consider a monochromatic rectangle which contains two singleton cells. In this case, the summation of weights in the monochromatic rectangle is 2. However, it is contained in the clique q . It implies that the limit of the total weight is decreased by 1. Thus, the constraint is satisfied. There are no monochromatic rectangles which contain more than 3 singleton cells because a rectangle which contains more than two kinds of singleton cells is not monochromatic. \square

Then, we generalize this idea for the case of the $(2l+1)$ -bit majority function.

Theorem 4.2.

$$L(\mathbf{MAJ}_{2l+1}) \geq (l+1)^2 + 1.$$

Proof. We consider a communication matrix of the majority function with $2l+1$ input bits whose rows and columns are restricted to minterms and maxterms, respectively. Let $m = \binom{2l+1}{l}$, which is equal to both the number of rows and columns. Then, the number of all cells is m^2 . The number of singleton cells is $(l+1)m$ and hence the number of singleton cells for each index is $\frac{(l+1)m}{2l+1}$. The number of cells with 3 indices is

$$\binom{l+1}{2} \cdot l \cdot m = \frac{l^2(l+1)m}{2}$$

because we can obtain a maxterm by flipping two bits of 1's to 0's and one bit of 0 to 1 for each minterm.

We consider 3×3 submatrices in the following way. From $2l+1$ input bits, we fix $2l-2$ arbitrary bits and assume that they have the same number of 0's and 1's. Then, we consider the remaining 3 bits. If the $2l+1$ input bits compose a minterm, the 3 bits are either 110, 101 or 011. If the $2l+1$ input bits compose a maxterm, the 3 bits are either 100, 010 or 001. Thus, we have a 3×3 submatrix, which has the same structure as the communication matrix of the 3-bit majority function as Figure 1. The number of submatrices is

$$\binom{2l+1}{3} \cdot \binom{2l-2}{l-1} = \frac{l^2(l+1)m}{6}.$$

Each submatrix has 6 singleton cells and 3 cells each of which has 3 indices corresponding to the remaining 3 bits. Note that each cell with 3 indices in any submatrix is not contained in other submatrices. In other words, all the $\frac{l^2(l+1)m}{2}$ cells with 3 indices are exactly partitioned into the $\frac{l^2(l+1)m}{6}$ submatrices.

We assign weights a for all singleton cells, 0 for cells with 3 indices and b for other cells, which have more than 3 indices. Note that there are no cells with 2 indices. We consider $\frac{l^2(l+1)m}{6}$ clique constraints assigned weights c (≤ 0) for all the $\frac{l^2(l+1)m}{6}$ submatrices. That is, we have a clique constraint for each submatrix similarly to the proof of Proposition 4.1. More precisely, a clique associated with a submatrix is composed of monochromatic rectangles which contain 2 singleton cells in the submatrix.

Then, the objective function of the dual problem is written as

$$\max_{a,b,c} (l+1)m \cdot a + \left(m^2 - (l+1)m - \frac{l^2(l+1)m}{2} \right) \cdot b + \frac{l^2(l+1)m}{6} \cdot c. \quad (1)$$

Now, we fix $c = 2b \leq 0$. Then, we have

$$\max_{a,b} (l+1)m \cdot a + \left(m^2 - (l+1)m - \frac{l^2(l+1)m}{6} \right) \cdot b. \quad (2)$$

We assume that a monochromatic rectangle contains k singleton cells and consider all possible pairs of 2 singleton cells taken from the k singleton cells. If a pair is in the same submatrix, the monochromatic rectangle is contained in a clique associated with the submatrix. If a pair is not in the same submatrix, the monochromatic rectangle contains two cells which are assigned weights b because they have more than 3 indices. Thus, if the following inequality is satisfied

$$k \cdot a + (k^2 - k) \cdot b \leq 1 \quad (3)$$

for any integer k $\left(1 \leq k \leq \frac{(l+1)m}{2l+1}\right)$, all constraints of the dual problem are satisfied when $c = 2b$.

We can maximize (2) by assuming that the inequality is saturated when

$$k = \frac{m}{l+1} - \frac{l^2}{6}$$

as it satisfies

$$\frac{k^2 - k}{k} = \frac{m^2 - (l+1)m - \frac{l^2(l+1)m}{6}}{(l+1)m}.$$

In this case, we have

$$(2) = \frac{(l+1)m}{\frac{m}{l+1} - \frac{l^2}{6}} = \frac{(l+1)^2 m}{m - \frac{1}{6}l^2(l+1)}$$

and obtain a lower bound

$$L(\mathbf{MAJ}_{2l+1}) \geq \frac{(l+1)^2}{1 - \epsilon(l)}$$

where $\epsilon(l) = \frac{l^2(l+1)}{6 \cdot \binom{2l+1}{l}}$. Since formula size must be an integer, we have shown the theorem. \square

5. Formula Size of Unbalanced Recursive Ternary Majority Functions

In this section, we show the following matching bound of formula size for unbalanced recursive ternary majority functions.

Theorem 5.1.

$$L(\mathbf{URecMAJ}_3^h) = L_m(\mathbf{URecMAJ}_3^h) = 4h + 1.$$

Proof. First, we look at the monotone formula size upper bound. Recall that a monotone formula of the 3-bit majority function can be written as $(x_1 \wedge x_2) \vee ((x_1 \vee x_2) \wedge x_3)$. The important point here is that the literal x_3 appears only once. We construct $(x_{2h} \wedge x_{2h+1}) \vee ((x_{2h} \vee x_{2h+1}) \wedge x_{2h-1})$ and replace x_{2h-1}

by a monotone formula representing $\mathbf{URecMAJ}_3^{h-1}$. A recursive construction yields a $4h + 1$ monotone formula for $\mathbf{URecMAJ}_3^h$.

We now show the non-monotone formula size lower bound. Before using clique constraints, we consider the original LP bound. The communication matrix of $\mathbf{URecMAJ}_3^h$ has some kind of recursive structure which can be informally stated as follows.

$$(\mathbf{URecMAJ}_3^h)^{-1}(1) := \{*\dots*11\} \cup \{(\mathbf{URecMAJ}_3^{h-1})^{-1}(1) + \{01 \mid 10\}\},$$

$$(\mathbf{URecMAJ}_3^h)^{-1}(0) := \{*\dots*00\} \cup \{(\mathbf{URecMAJ}_3^{h-1})^{-1}(0) + \{01 \mid 10\}\}.$$

We can interpret it in the following recursive way as Figure 3. In the figure,

	$(\dots)00$	$(\dots)10$	$(\dots)01$
$(\dots)11$	$[\dots]$	Y_{2h+1}	Y_{2h}
$(\dots)01$	X_{2h+1}	$[\dots]$	S_h
$(\dots)10$	X_{2h}	S'_h	$[\dots]$

Figure 3: Recursive Structure of the Communication Matrix of $\mathbf{URecMAJ}_3^h$ ($h \geq 2$)

rows denoted by “ $(\dots)11$ ”, “ $(\dots)01$ ” and “ $(\dots)10$ ” means sets of inputs in $(\mathbf{URecMAJ}_3^h)^{-1}(1)$ which have 11, 01 and 10 in the $2h$ -th and $(2h+1)$ -th bits, respectively. Similarly, columns denoted by “ $(\dots)00$ ”, “ $(\dots)10$ ” and “ $(\dots)01$ ” means sets of inputs in $(\mathbf{URecMAJ}_3^h)^{-1}(0)$ which have 00, 10 and 01 in the $2h$ -th and $(2h+1)$ -th bits, respectively.

Inside the communication matrix of $\mathbf{URecMAJ}_3^h$, we consider the following two submatrices denoted by S_h and S'_h in Figure 3

$$S_h := \{(\mathbf{URecMAJ}_3^{h-1})^{-1}(1) + 01\} \times \{(\mathbf{URecMAJ}_3^{h-1})^{-1}(0) + 01\},$$

$$S'_h := \{(\mathbf{URecMAJ}_3^{h-1})^{-1}(1) + 10\} \times \{(\mathbf{URecMAJ}_3^{h-1})^{-1}(0) + 10\}$$

in which any cell contains indices of neither $2h$ nor $2h+1$. So they have the same structure as the communication matrix of $\mathbf{URecMAJ}_3^{h-1}$.

All the three kinds of singleton cells $\{1\}$, $\{2\}$ and $\{3\}$ are included in either S_h or S'_h because all the other cells outside S_h and S'_h contain at least an index $2h$ or $2h+1$. There are no singleton cells in diagonal submatrices denoted by $[\dots]$ in Figure 3. Since each of S_h and S'_h contains two submatrices S_{h-1} and S'_{h-1} , the number of singleton cells $\{1\}$, $\{2\}$ and $\{3\}$ in S_h and S'_h doubles with each iteration. So, the total number of singleton cells $\{1\}$, $\{2\}$ and $\{3\}$ is $3 \cdot 2^h$.

Then, we consider the minimum submatrix M_h which contains all of three kinds of singleton cells $\{1\}$, $\{2\}$ and $\{3\}$. In other words, it is the submatrix spanned by all singleton cells $\{1\}$, $\{2\}$ and $\{3\}$. It does not contain any other kinds of singleton cells except $\{1\}$, $\{2\}$ and $\{3\}$. A submatrix M_1 is the communication matrix of the 3-bit majority function restricted to minterms and maxterms as shown in Figure 1. Both the number of rows and columns of M_h is equal to $3 \cdot 2^{h-1}$ because M_h 's duplicate $(h-1)$ -times from M_1 and does not

have any common rows and columns. Hence, the number of all cells in M_h is $9 \cdot 4^{h-1}$.

We assign weights a for all the singleton cells in M_h and weights b for all other cells in M_h . Then, the total weight of all cells in M_h is written as follows.

$$\max_{a,b} 3 \cdot 2^h \cdot a + (9 \cdot 4^{h-1} - 3 \cdot 2^h) \cdot b. \quad (4)$$

We consider constraints of the dual problem as $k \cdot a + (k^2 - k) \cdot b \leq 1$ for all integer k ($1 \leq k \leq 2^h$). We assume this inequality is saturated if and only if $k = 3 \cdot 2^{h-2}$. Then, we get $a = \frac{24 \cdot 2^h - 16}{9 \cdot 4^h}$ and $b = -\frac{16}{9 \cdot 4^h}$. In this case, (4) = 4.

Next, we consider singleton cells $\{2l\}$ and $\{2l+1\}$ ($2 \leq l \leq h$) in X_{2l} , X_{2l+1} , Y_{2l} and Y_{2l+1} . As shown in Figure 3, they are determined in the following way.

$$\begin{aligned} X_{2l} &:= \{(\mathbf{URecMAJ}_3^{l-1})^{-1}(1) + 10\} \times \{*\cdots*00\}, \\ X_{2l+1} &:= \{(\mathbf{URecMAJ}_3^{l-1})^{-1}(1) + 01\} \times \{*\cdots*00\}, \\ Y_{2l} &:= \{*\cdots*11\} \times \{(\mathbf{URecMAJ}_3^{l-1})^{-1}(0) + 01\}, \\ Y_{2l+1} &:= \{*\cdots*11\} \times \{(\mathbf{URecMAJ}_3^{l-1})^{-1}(0) + 10\}. \end{aligned}$$

A submatrix spanned by $X_{2l} \cup X_{2l+1}$ and $Y_{2l} \cup Y_{2l+1}$ dominates M_l in the sense that any rectangle containing all cells in $X_{2l} \cup X_{2l+1}$ and $Y_{2l} \cup Y_{2l+1}$ also contains all cells in M_l . Therefore, we can restrict these sets to the minimum subsets

$$X'_{2l} \subseteq X_{2l}, X'_{2l+1} \subseteq X_{2l+1}, Y'_{2l} \subseteq Y_{2l}, Y'_{2l+1} \subseteq Y_{2l+1}$$

so as to satisfy that a submatrix spanned by $X'_{2l} \cup X'_{2l+1}$ and $Y'_{2l} \cup Y'_{2l+1}$ is the minimum submatrix dominating M_l .

Similarly to M_l , each of X'_{2l} , X'_{2l+1} , Y'_{2l} and Y'_{2l+1} also duplicates $(h-l)$ -times in the communication matrix of $\mathbf{URecMAJ}_3^h$ (not $\mathbf{URecMAJ}_3^l$). We take unions of these duplicated sets as X''_{2l} , X''_{2l+1} , Y''_{2l} and Y''_{2l+1} , respectively. The number of rows and columns for each of X'_{2l} , X'_{2l+1} , Y'_{2l} and Y'_{2l+1} is $3 \cdot 2^{h-2}$, which is the half of the number corresponding to M_h because a submatrix spanned by $X''_{2l} \cup X''_{2l+1}$ and $Y''_{2l} \cup Y''_{2l+1}$ tightly dominates M_h . For each row and each column of $X''_{2l} \cup X''_{2l+1}$ and $Y''_{2l} \cup Y''_{2l+1}$, there is exactly one singleton cell, which is either $\{2l\}$ or $\{2l+1\}$ according to the subscript, and no other kinds of singleton cells. Thus, the number of singleton cells for each of X''_{2l} , X''_{2l+1} , Y''_{2l} and Y''_{2l+1} is $3 \cdot 2^{h-2}$.

For each l ($2 \leq l \leq h$), we assign weights $\frac{1}{3 \cdot 2^{h-2}}$ for all the singleton cells $\{2l\}$ and $\{2l+1\}$ in $X''_{2l} \cup X''_{2l+1}$ and $X''_{2l+1} \cup Y''_{2l+1}$, respectively, and 0 for all the other cells outside M_h . A monochromatic rectangle which contains x cells in X''_{2l} and y cells in Y''_{2l} also contains $x \cdot y$ cells in M_h and submatrices denoted by $[\cdots]$ in Figure 3. These $x \cdot y$ cells in M_h have been already assigned weights $b = -\frac{16}{9 \cdot 4^h}$. The same thing is true for the case of X''_{2l+1} and Y''_{2l+1} . Since we

have

$$(x + y) \cdot \frac{4}{3 \cdot 2^h} - xy \cdot \frac{16}{9 \cdot 4^h} \leq 1 \quad (5)$$

for all $0 \leq x, y \leq 3 \cdot 2^{h-2}$, all the constraints without clique variables of the dual problem are satisfied. The total weight of singleton cells $\{2l\}$ and $\{2l+1\}$ is 4. So, the total weight of all cells now becomes $4h$.

Now, we add the clique constraints to the primal problem and the adjust the dual variables accordingly by moving some of the weight from the cell variables to the clique variables. Here each of the cliques given for each M_1 is the same one argued in the proof of Proposition 4.1. The number of M_1 in M_h is 2^{h-1} . We change weights of all non-singleton cells in submatrices S_1 from b to 0. On behalf of them, we consider a clique variable of the dual problem for each S_1 in S_h and change weights of the clique variables from 0 to $c = 2b$. Then, (4) becomes

$$\max_{a,b,c} 3 \cdot 2^h \cdot a + (9 \cdot 4^{h-1} - 3 \cdot 2^h - 3 \cdot 2^{h-1}) \cdot b + 2^{h-1} \cdot c. \quad (6)$$

If we take $a = \frac{24 \cdot 2^h - 16}{9 \cdot 4^h}$, $b = -\frac{16}{9 \cdot 4^h}$ and $c = 2b$, all the constraints of the dual problem are satisfied and

$$(6) = 4 + \frac{8}{9} \cdot 2^{-h}.$$

Consequently, the objective value is $4h + \frac{8}{9} \cdot 2^{-h}$. Since formula size must be an integer, we have shown the theorem. \square

6. Monotone Formula Size of Balanced Recursive Ternary Majority Functions

In this section, we show monotone formula size lower bounds of balanced recursive ternary majority functions. For this purpose, we consider rank constraints, which are generalizations of clique constraints. Similarly to the case of clique constraints, we consider a graph G composed of monochromatic rectangles and its induced subgraph H . Let $\alpha(H)$ be the stability number of H , the maximum number of vertices, no two of which are adjacent. If $\alpha(H) = 1$, then H is a clique. We consider a constraint

$$\sum_{r \in H} x_r \leq \alpha(H).$$

where $r \in H$ means the vertex corresponding to a rectangle r is contained in the induced subgraph H .

The rank constraint is valid for any induced subgraph H in the following reason. Recall that in the induced graph, vertices represent rectangles and an edge is included if the two rectangles intersect. Since the rectangles in a

partition of the matrix are disjoint, the subgraph induced by a disjoint cover form an independent set. Hence in any integral feasible solution, the number of rectangles cannot exceed the size of the largest independent set. In other words, we can assign 1 at most $\alpha(H)$ rectangles in H for any integral solution.

The dual problem can be written as follows.

$$\begin{aligned} \max \quad & \sum_c w_c + \sum_q z_q + \sum_H \alpha(H) z_H \\ \text{s.t.} \quad & \sum_{c \in r} w_c + \sum_{q \ni r} z_q + \sum_{H \ni r} z_H \leq 1, \quad (\text{for each monochromatic rectangle } r) \\ & z_q \leq 0, \quad (\text{for each clique } q) \\ & z_H \leq 0. \quad (\text{for each subgraph } H) \end{aligned}$$

Before going to the general case, we consider the case of height 2. By using clique constraints and rank constraints, we prove the following improved monotone formula size lower bound while we know that the original LP bound cannot prove a lower bound larger than 16.5. It is easy to verify this by showing an upper bound (fractional partition) for the primal formulation for the LP bound.

Proposition 6.1.

$$L_m(\mathbf{BRecMAJ}_3^2) \geq 20.$$

Proof. There are 27 minterms and 27 maxterms for the recursive ternary majority function of height 2. Among them, we choose the following 9 minterms

$$\begin{array}{lll} 110,110,000 & 101,101,000 & 011,011,000 \\ 110,000,110 & 101,000,101 & 011,000,011 \\ 000,110,110 & 000,101,101 & 000,011,011 \end{array}$$

and 9 maxterms

$$\begin{array}{lll} 111,100,100 & 111,010,010 & 111,001,001 \\ 100,111,100 & 010,111,010 & 001,111,001 \\ 100,100,111 & 010,010,111 & 001,001,111. \end{array}$$

From these 9 minterms and 9 maxterms, a submatrix of the communication matrix can be described as Figure 4. In the figure, we abbreviate a minterm e.g. 101,101,000 by 110 and 101, which represent the second level and the first level structure of the 9 bits, respectively. Notice that all minterms which we choose have the same structure in all 3-bit minterm blocks at the first level. The same thing is true for all 9 maxterms.

To describe 12 cliques q_1, \dots, q_{12} and a induced subgraph H whose stability number is 4, we give serial numbers for 81 cells as Figure 5. We take the following 12 cliques each of which consists of 3 pairs of 2 singleton cells.

$$\begin{aligned} & \{ (5, 15), (4, 24), (13, 23) \}, \{ (35, 45), (34, 54), (43, 53) \}, \\ & \{ (2, 12), (1, 21), (10, 20) \}, \{ (62, 72), (61, 81), (70, 80) \}, \\ & \{ (29, 39), (28, 48), (37, 47) \}, \{ (59, 69), (58, 78), (67, 77) \}, \\ & \{ (5, 35), (2, 62), (29, 59) \}, \{ (15, 45), (12, 72), (39, 69) \}, \\ & \{ (4, 34), (1, 61), (28, 58) \}, \{ (24, 54), (21, 81), (48, 78) \}, \\ & \{ (13, 43), (10, 70), (37, 67) \}, \{ (23, 53), (20, 80), (47, 77) \}. \end{aligned}$$

		100			010			001		
		100	010	001	100	010	001	100	010	001
110	110	5	4	4,5	2	1	1,2	2,5	1,4	1,2,4,5
	101	6	4,6	4	3	1,3	1	3,6	1,3,4,6	1,4
	011	5,6	6	5	2,3	3	2	2,3,5,6	3,6	2,5
101	110	8	7	7,8	2,8	1,7	1,2,7,8	2	1	1,2
	101	9	7,9	7	3,9	1,3,7,9	1,7	3	1,3	1
	011	8,9	9	8	2,3,8,9	3,9	2,8	2,3	3	2
011	110	5,8	4,7	4,5,7,8	8	7	7,8	5	4	4,5
	101	6,9	4,6,7,9	4,7	9	7,9	7	6	4,6	4
	011	5,6,8,9	6,9	5,8	8,9	9	8	5,6	6	5

Figure 4: A Submatrix of the Communication Matrix of $\mathbf{BRecMAJ}_3^2$

		100			010			001		
		100	010	001	100	010	001	100	010	001
110	110	1	2	3	4	5	6	7	8	9
	101	10	11	12	13	14	15	16	17	18
	011	19	20	21	22	23	24	25	26	27
101	110	28	29	30	31	32	33	34	35	36
	101	37	38	39	40	41	42	43	44	45
	011	46	47	48	49	50	51	52	53	54
011	110	55	56	57	58	59	60	61	62	63
	101	64	65	66	67	68	69	70	71	72
	011	73	74	75	76	77	78	79	80	81

Figure 5: Serial Numbers for 81 cells of the Submatrix

For each combination of 3 pairs in 12 cliques, it is easy to verify that rectangles each of which contains both of 2 singleton cells from one of the 3 pairs compose a clique. For each 2 clique of each line, they are concerned with the following triplet of indices, respectively.

$$\{1,2,3\}, \{4,5,6\}, \{7,8,9\}, \{1,4,7\}, \{2,5,8\}, \{3,6,9\}$$

The first 3 triplets of indices capture the first level structure of the recursion in some sense. A similar thing is true for the last 3 triplets of indices which capture the second level structure of the recursion.

To get a better lower bound, we would like to utilize more combinations of indices. In this regard, a rank constraint is very useful while clique constraints are powerless anymore. We consider the following 18 pairs of singleton cells which induce the subgraph H .

$$(5, 45), (15, 35), (4, 54), (24, 34), (13, 53), (23, 43),$$

$$(2, 72), (12, 62), (1, 81), (21, 61), (10, 80), (20, 70), \\ (29, 69), (39, 59), (28, 78), (48, 58), (37, 77), (47, 67).$$

If a rectangle contains both of two singleton cells from one of 18 pairs, it also contains 2 cells from 9 cells $\{ 9, 17, 25, 33, 41, 49, 57, 65, 73 \}$. Thus, we can choose at most 4 pairs without conflicts from 18 pairs. It implies that the stability number of H is 4.

Notice that all these 12 cliques and the subgraph cover all pairs of two singleton cells which have the same index. We assign 1 for all 36 singleton cells in this submatrix and 0 for other cells. We take $z_{q_1} = \dots = z_{q_{12}} = z_H = -1$. Then, the objective value of the dual problem becomes $36 - 12 - 4 = 20$. If a rectangle contains at most one singleton cell, the constraint of the dual problem is trivially satisfied. If a rectangle contains k ($2 \leq k \leq 4$) singleton cells, it is covered by $k - 1$ cliques or $k - 2$ cliques plus the subgraph H . So, the constraint is also satisfied. As a consequence, we obtain the formula size lower bound. \square

Note that we need a much more complicated argument to look at the non-monotone case, which we do not investigate in this paper, because singleton cells in the monotone communication matrix are not singleton in the non-monotone communication matrix.

In the general monotone case, we can prove a slightly better lower bound than the quantum adversary bound [15], which shows a 4^h lower bound.

Theorem 6.2. *For $h \geq 2$,*

$$L_m(\mathbf{BRecMAJ}_3^h) \geq 4^h + \frac{13}{36} \cdot \left(\frac{8}{3}\right)^h.$$

Proof. First, we choose 3^h minterms and 3^h maxterms from 3^h input bits of $\mathbf{BRecMAJ}_3^h$ so as to have the same structure in the 1st, 2nd, \dots and h -th levels in the following sense. In the l -th level, we have 3^{h-l} bits which are recursively constructed from lower levels in the following way. We partition 3^l bits into 3^{l-1} blocks each of which contains consecutive 3 bits. For each block of 3 bits, we replace them into 1 bit which is the output of \mathbf{MAJ}_3 with the 3 bits. Then, we get $3^{h-(l+1)}$ bits. We have 3^h bits as input bits in the first level and can construct them for each level by induction. If all of 3^{l-1} blocks have the same 3 bits except 000 and 111 in the case of minterms and maxterms, respectively, we call that they have the same structure in the l -th level. There are 3^h minterms and 3^h maxterms because we have 3 choices in each level. We consider the submatrix whose rows and columns are composed of these 3^h minterms and 3^h maxterms, respectively.

From another viewpoint, we can interpret it as a recursively construction of the submatrix S_h of the communication matrix of $\mathbf{BRecMAJ}_3^h$ as follows. We define $S_h(k)$ ($k = 1, 2, 3$) as a matrix such that some cell of $S_h(k)$ contains an index $(k-1) \cdot 3^h + i$ if and only if the corresponding cell of S_h contains an index i . By induction, we can see that the number of all cells and singleton cells in S_h is 9^h and 6^h , respectively. Singleton cells of each index from 3^h bits in S_h is 2^h .

Indices of cells in $T_h(1, 2)$, $T_h(2, 3)$ and $T_h(2, 3)$ in Figure 6 can be determined from the property of combinatorial rectangles, but we do not go to the details because we will assign the same weight for all these cells in each level.

	100	010	001
110	$S_{h-1}(2)$	$S_{h-1}(1)$	$T_{h-1}(1, 2)$
101	$S_{h-1}(3)$	$T_{h-1}(2, 3)$	$S_{h-1}(1)$
011	$T_{h-1}(2, 3)$	$S_{h-1}(3)$	$S_{h-1}(2)$

Figure 6: Recursive Structure of S_h for $\mathbf{BRecMAJ}_3^h$ ($h \geq 2$)

Before using clique and rank constraints, we consider the original LP bound. We assign weights a for all singleton cells, b for other cells in the submatrix and 0 for all cells in the outside of the submatrix. Then, the objective value of the dual problem is written as

$$\max_{a,b} 6^h \cdot a + (9^h - 6^h) \cdot b. \quad (7)$$

If a rectangle contains k singleton cells, it also contains at least $k^2 - k$ cells which are not singleton. Thus, if $k \cdot a + (k^2 - k) \cdot b \leq 1$ is satisfied for all integer k ($1 \leq k \leq 2^h$), then all constraints of the dual problem are also satisfied. We assume that the inequality is saturated if and only if $k = (3/2)^h$. Then, we get $a = \frac{2 \cdot 6^h - 4^h}{9^h}$ and $b = -\frac{4^h}{9^h}$. In this case, we have $(7) = 4^h$.

Now, we incorporate clique and rank constraints. We change weights of all cells except singleton cells in all S_2 's in the second level from b to 0. Then, we add 12 clique constraints and a rank constraint for each S_2 in the second level by following the way of Proposition 6.1. Let c and d be values assigned for every clique and rank constraints, respectively. Then, the objective value of the dual problem is

$$\max_{a,b,c,d} 6^h \cdot a + (9^h - 81 \cdot 6^{h-2}) \cdot b + 12 \cdot 6^{h-2} \cdot c + 4 \cdot 6^{h-2} \cdot d. \quad (8)$$

If we take $c = d = 2b$, then we have

$$(8) = 6^h \cdot a + (9^h - 49 \cdot 6^{h-2}) \cdot b = 4^h + \frac{13}{36} \cdot \left(\frac{8}{3}\right)^h.$$

Since all weights which are changed from b to 0 are exactly compensated by clique and rank constraints, all constraints of the dual problem are satisfied. \square

We do not exhaust the potential of our new method and have possibilities to improve the lower bound. For example, we can improve the lower bound as $4^h + c \cdot \left(\frac{8}{3}\right)^h$ for some constant c by further detailed analysis in constantly higher levels such as $\mathbf{BRecMAJ}_3^3$, $\mathbf{BRecMAJ}_3^4$ and so on.

7. Conclusions

In this paper, we devised the new technique proving formula size lower bounds and showed improved formula size lower bounds of some families of monotone self-dual Boolean functions such as majority functions, unbalanced and balanced recursive ternary majority functions. We hope that our method will be able to improve formula size lower bounds for any monotone self-dual Boolean function and even much broader classes of Boolean functions. Whether our technique (or its extensions) can break the $4n^2$ barrier and improve the lower bounds remains open.

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